of analysis, can be used to show the existence of various implicitly defined functions.

For a simple illustration of the content, power, and use of the recursion theorem, consider the following question. Does there exist an m such that $W_m = \{m\}$? At first glance, the existence of such an m might appear to be an arbitrary and accidental feature of the indexing of recursively enumerable sets. Indeed, it might seem likely that, in our indexing, no such m exists. In §11.2, however, we use the recursion theorem to show that such an m must exist for our indexing of recursively enumerable sets and for all other acceptable indexings of the recursively enumerable sets.

The recursion theorem is due to Kleene (as are many of its known applications).

§11.2 THE RECURSION THEOREM

The recursion theorem is given in its strongest and most general form in Theorem IV below. For expository purposes, three simpler versions of the theorem are given first. The results stated explicitly in Theorems II, III, and IV are already implicit in the proof for Theorem I.

Theorem I *Let f be any recursive function; then there exists an n such that*

$$\varphi_n = \varphi_{f(n)}.$$

(We call n a *fixed-point value* for f.)

Proof. Let any u be given. Define a recursive function ψ by the following instructions: to compute $\psi(x)$, first use P_u with input u; if and when this terminates and gives an output w, use P_w with input x; if and when this terminates, take its output as $\psi(x)$. We summarize this:

$$\psi(x) = \begin{cases} \varphi_{\varphi_u(u)}(x), & \text{if } \varphi_u(u) \text{ convergent;} \\ \text{divergent,} & \text{if } \varphi_u(u) \text{ divergent.} \end{cases}$$

The instructions for ψ depend uniformly on u. Take $\tilde{g}$ to be the recursive function which yields, from u, the Gödel number for these instructions for ψ. Thus

$$\varphi_{\tilde{g}(u)}(x) = \begin{cases} \varphi_{\varphi_u(u)}(x), & \text{if } \varphi_u(u) \text{ convergent;} \\ \text{divergent,} & \text{if } \varphi_u(u) \text{ divergent.} \end{cases}$$

Now let any recursive function f be given. Then $f\tilde{g}$ is a recursive function. Let v be a Gödel number for $f\tilde{g}$. Since $\varphi_v = f\tilde{g}$ is total, $\varphi_v(v)$ is convergent. Hence, putting v for u in the definition of $\tilde{g}$, we have

$$\varphi_{\tilde{g}(v)} = \varphi_{\varphi_v(v)} = \varphi_{f\tilde{g}(v)}.$$

Thus $n = \tilde{g}(v)$ is a fixed-point value, as desired. $\boxtimes$

Corollary I *Let f be any recursive function; then there exists an n such that $W_n = W_{f(n)}$.*

Proof. Immediate. $\boxtimes$

Example. Consider the illustrative question suggested in §11.1: *does there exist an m such that $W_m = \{m\}$?* By Church's Thesis (and the *s-m-n* theorem), there is a recursive function f such that for any x, $W_{f(x)} = \{x\}$. Applying Corollary I, we have an n such that $W_n = W_{f(n)}$. Hence $W_n = \{n\}$, and the question asked in §11.1 has an affirmative answer. (Note that this proof can be carried through for any Gödel numbering that is *acceptable* in the sense of Exercise 2-10; see also Exercise 11-12.)

Theorem I can be strengthened in three ways: (1) we can show that n depends uniformly on a Gödel number for f; (2) we can show that when f involves other parameters recursively, then n can be made to depend uniformly on those parameters; and (3) we can show that for any f, an infinite set of fixed-point values can be recursively enumerated. We give (1) and (2) separately in Theorems II and III. We combine (1), (2), and (3) in our most general formulation, Theorem IV. Theorems I, II, and III are all included in Theorem IV. Theorems I and III will be the forms most commonly used in applications.

Theorem II *There exists a recursive function n such that for any z, if φ_z is total, then*

$$\varphi_{n(z)} = \varphi_{\varphi_z(n(z))}.$$

Proof. Let $\varphi_z = f$ and consider the proof of Theorem I. By Theorem 1-VI, v, the Gödel number for $f\tilde{g}$, can be obtained uniformly from z. Let $\tilde{v}$ be a recursive function such that $\varphi_{\tilde{v}(z)} = \varphi_z\tilde{g}$. Then our desired recursive function n is obtained by defining $n(z) = \tilde{g}\tilde{v}(z)$. $\boxtimes$

Corollary II *There exists a recursive function n such that for any z, if φ_z is total, then*

$$W_{n(z)} = W_{\varphi_z(n(z))}.$$

Proof. Immediate. $\boxtimes$

Theorem III *Let f be a recursive function of $k+1$ variables. Then there exists a recursive function n_f of k variables such that for all $x_1, \ldots, x_k$,*

$$\varphi_{n_f(x_1,\ldots,x_k)} = \varphi_{f(n_f(x_1,\ldots,x_k),x_1,\ldots,x_k)}.$$

Proof. The construction parallels that in Theorem I. Define $\tilde{g}$ to be a function of $k+1$ variables such that

$$\varphi_{\tilde{g}(u,x_1,\ldots,x_k)}(y) = \begin{cases} \varphi_{\varphi_u^{(k+1)}(u,x_1,\ldots,x_k)}(y), & \text{if } \varphi_u^{(k+1)}(u,x_1,\ldots,x_k) \text{ is convergent;} \\ \text{divergent,} & \text{if } \varphi_u^{(k+1)}(u,x_1,\ldots,x_k) \text{ is divergent.} \end{cases}$$